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Scaling function in AdS/CFT from the O(6) sigma model

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Abstract

Asymptotic behavior of the anomalous dimensions of Wilson operators with high spin and twist is governed in planar $\mathcal{N}=4$ SYM theory by the scaling function which coincides at strong coupling with the energy density of a two-dimensional bosonic O(6) sigma model. We calculate this function by combining the two-loop correction to the energy density for the O(n) model with two-loop correction to the mass gap determined by the all-loop Bethe ansatz in $\mathcal{N}=4$ SYM theory. The result is in agreement with the prediction coming from the thermodynamical limit of the quantum string Bethe ansatz equations, but disagrees with the two-loop stringy corrections to the folded spinning string solution.

1 Introduction

The AdS/CFT [1] establishes the correspondence between Wilson operators in $\mathcal{N} = 4$ super Yang-Mills theory (SYM) and states of strings spinning on $\mathrm{AdS}_5 \times \mathrm{S}^5$ background [2, 3]. It has been recently recognized that there exists a remarkable relation between both theories and the two-dimensional bosonic O(6) sigma model. This relation emerges when one studies anomalous dimensions of Wilson operators in planar $\mathcal{N} = 4$ SYM theory at strong coupling in the limit [4] when the Lorentz spin N of the operators grows exponentially with their twist L

$$j = \frac{L}{\ln N} = \text{fixed}\,, \qquad\qquad N,\ L \to \infty\,. \tag{1.1}$$

In this limit, the anomalous dimensions grow logarithmically with N and their leading asymptotic behavior is governed by the scaling function[1] depending on j and the t' Hooft coupling $g^2 = g_{\rm YM}^2 N_c/(4\pi)^2$. Using the dual description of Wilson operators as folded strings spinning on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and taking into account the one-loop stringy corrections to these states [5], Alday and Maldacena [6] conjectured that the scaling function should coincide at strong coupling with the energy density of a two-dimensional bosonic O(6) sigma model. Recently, this relation was established in planar $\mathcal{N} = 4$ SYM theory at strong coupling [7] using the conjectured integrability of the dilatation operator [8].

The O(6) sigma-model and, in general, two-dimensional bosonic O(n) sigma-models are among the best studied field theory models. Their popularity is explained by the fact that they can be used as low dimensional toy models of QCD: they are asymptotically free in perturbation theory, their classical conformal invariance is broken and the mass of the physical excitations is dynamically generated by the dimensional transmutation mechanism [9, 10]. In addition they are integrable and many physical quantities are exactly calculable [11]. These models can also be studied non-perturbatively with a help of lattice Monte Carlo simulations making use of a very efficient simulation algorithm (the cluster algorithm).

Using the known exact S-matrix of the O(n) sigma-model, a linear thermodynamical Bethe ansatz (TBA) integral equation can be derived describing the free energy in the presence of an external field coupled to one of the Noether currents of the model.[2] The free energy can also be calculated in perturbation theory for large values of the external field due to asymptotic freedom. The original motivation of this calculation [12] was that by comparing the two results for the free energy one can calculate the physical mass m of the O(n) particles in terms of the perturbative Λ-parameter (dynamical scale defining solutions to Gell-Mann-Low equation). The exact mass gap m/Λ determined this way has been checked by using Monte Carlo results and also by comparing finite volume mass gap values computed in perturbation theory [13] to those obtained from a (nonlinear) TBA integral equation [14].

In this paper, we employ tools developed for the O(n) sigma-model to compute the scaling function in the AdS/CFT.

1.1 Scaling function in $\mathcal{N} = 4$ SYM

The Wilson operators under consideration are built from L complex scalar fields and N light-cone components of the covariant derivatives. Their minimal anomalous dimension has the following

[1] More precisely, for given spin N and twist L, anomalous dimensions of Wilson operators occupy a band. The scaling function describes the *minimal* anomalous dimension in this band.

[2] The energy density is related to the free energy through Legendre transformation.

behavior both at weak and at strong coupling [4, 6, 8, 5, 15]

$$\gamma_{N,L}(g) = [2\Gamma_{\rm cusp}(g) + \epsilon(g,j)] \ln N + \dots\,, \tag{1.2}$$

where ellipses denote terms suppressed by powers of $1/L$. Here the first term inside the square brackets has a universal, $j-$independent form and it involves the cusp anomalous dimension [16, 17]. The dependence on the twist resides in $\epsilon(g,j)$ which is a nontrivial function of the 't Hooft coupling and the scaling variable j normalized as $\epsilon(g,0)=0$.

At weak coupling, the scaling functions $\Gamma_{\rm cusp}(g)$ and $\epsilon(g,j)$ can be found in a generic (supersymmetric) Yang-Mills theory in the planar limit by making use of the remarkable property of integrability [4, 17]. In maximally supersymmetric $\mathcal{N}=4$ theory, these functions can be determined in the planar limit for arbitrary values of the scaling parameter j and the coupling g as solutions to BES/FRS equations proposed in [8, 18]. These equations predict the scaling function to be a bi-analytical function of g^2 and j. [3] At strong coupling, the asymptotic expansion of the cusp anomalous dimension in powers of $1/g$ was derived in [20, 21, 22, 23, 24]. It turned out that this expansion suffers from Borel singularities [24] indicating that $\Gamma_{\rm cusp}(g)$ receives nonperturbative corrections at strong coupling defined by the scale $m \sim g^{1/4}\,{\rm e}^{-\pi g}$ [6, 24].

At strong coupling, the scaling function $\epsilon(g,j)$ has a more complicated form and its properties depend on a hierarchy between g and j. As was shown in [7], for $g\to\infty$ and $j/m=$ fixed (with m given by (1.4) below), the FRS equation for the scaling function $\epsilon(g,j)$ can be casted into a form identical to the TBA equations for the nonlinear O(6) sigma model [12]. This leads to the identification of the scaling function as the energy density $\varepsilon_{\rm O(6)}$ in the ground state of the O(6) model corresponding to the particle density ρ

$$\varepsilon_{\rm O(6)} = \frac{\epsilon(g,j)+j}{2}\,, \qquad\qquad \rho = \frac{j}{2}\,. \tag{1.3}$$

As was already mentioned, the O(6) sigma model has a nontrivial dynamics in the infrared and it develops a mass gap. Remarkably enough, similar phenomenon also occurs for the scaling function in $\mathcal{N}=4$ theory at strong coupling. Namely, the scaling function $\epsilon(g,j)$ depends on a new dynamical scale [7, 25]

$$m = k g^{1/4}\,{\rm e}^{-\pi g}\,[1+O(1/g)]\,, \qquad\qquad k = \frac{2^{3/4}\pi^{1/4}}{\Gamma(5/4)}\,, \tag{1.4}$$

and the same scale (1.4) defines nonperturbative corrections to the cusp anomalous dimension at strong coupling [24, 26]. Later in the paper, we will compute subleading corrections to the mass scale (1.4). The relation (1.3) holds for $g\to\infty$ with $j/m=$ fixed and the scale (1.4) is identified as the mass gap of the O(6) model.

1.2 Scaling function in AdS/CFT

The AdS/CFT correspondence relates the anomalous dimension (1.2) at strong coupling to the energy of a folded string spinning on the $\rm AdS_5\times S^5$ background [2, 3]

$$\Delta = N + L + \gamma_{N,L}(g)\,, \tag{1.5}$$

[3] The properties of this function at weak coupling and large j were studied in [19].

with N and L being angular momenta on AdS_3 and S^1, respectively. Semiclassical quantization of this stringy state yields the expansion of the anomalous dimension $\gamma_{N,L}(g)$ in powers of $1/g$. In agreement with (1.2), the coefficients of the expansion scale logarithmically with N in the limit (1.1) and determine stringy corrections to the scaling function.

For the cusp anomalous dimension $\Gamma_{\mathrm{cusp}}(g)$, the first three coefficients of $1/g$ expansion were computed in Refs. [2, 5, 27]. As a nontrivial test of the AdS/CFT correspondence, they were found to be the same as in the $\mathcal{N} = 4$ SYM theory [24].[4] For the scaling function similar calculation was performed by two different approaches – from two-loop stringy corrections to the folded spinning string solution [28] and from thermodynamical limit of quantum string Bethe ansatz equations [15, 29, 30], leading to (in notations of [28])

$$\epsilon(g,j)+j = 2\ell^2\left[g+\frac{1}{\pi}\left(\frac{3}{4}-\ln\ell\right)+\frac{1}{4\pi^2 g}\left(\frac{q_{02}}{2}-3\ln\ell+4(\ln\ell)^2\right)+\mathcal{O}\left(1/g^2\right)\right]+O(\ell^4)\,, \quad (1.6)$$

with $\ell = j/(4g)$. This relation is valid at strong coupling for $j \ll g$, or equivalently $\ell \ll 1$. Here the first two terms inside the square brackets describe, correspondingly, the classical expression and one-loop correction to the scaling function $\epsilon(g,j)$. The last term describes the two-loop correction and it depends on a constant q_{02}. The two approaches mentioned above produce two different values of q_{02}

$$q_{02}\Big|_{\text{ref.[28]}} = -2\mathrm{K}-\frac{3}{2}\ln 2+\frac{7}{4}\,, \qquad q_{02}\Big|_{\text{ref.[30]}} = -\frac{3}{2}\ln 2+\frac{11}{4}\,, \quad (1.7)$$

with K being the Catalan constant. The two results agree with each other in term $\sim \ln 2$ but disagree in the rest. The reason for this discrepancy remains unclear.

The semiclassical approach allows us to calculate the scaling function (1.6) in the form of a double series in $1/g$ and ℓ^2. It does not take however into account nonperturbative corrections to the scaling function which are exponentially small as $g \to \infty$. Alday and Maldacena [6] put forward the proposal that the scaling function $\epsilon(g,j)$ can be found *exactly* at strong coupling in the limit $j \ll g$ and $j/m =$ fixed (with m defined in (1.4)). They argued that quantum corrections in the $\mathrm{AdS}_5\times\mathrm{S}^5$ sigma model are dominated in this limit by the contribution of massless excitations on S^5 whose dynamics is described by a (noncritical) two-dimensional bosonic O(6) sigma-model equipped with a UV cut-off determined by the mass of massive excitations. In terms of parameters of the underlying $\mathrm{AdS}_5\times\mathrm{S}^5$ sigma model, the exact value of the mass gap is given by (1.4). The dependence of the mass scale (1.4) on the coupling constant is fixed by the two-loop beta-function of the O(6) model whereas the prefactor k was determined in [6] by matching the first two terms of the semiclassical expansion (1.6) into known one-loop perturbative correction to the energy density of the O(6) model.

The subleading $O(1/g)$ correction to the scaling function (1.6) involves both constant term q_{02} and logarithmically enhanced terms. As was shown in [6], the latter terms are controlled by renormalization group. It is the constant q_{02} that lies at crux of the relation (1.6) to two-loop order. The relation (1.3) allows us to determine this constant by computing corrections to the energy density of the O(6) model and by matching the resulting expression for the scaling function into (1.6).

[4]Note that the semiclassical approach does not take into account nonperturbative corrections to $\Gamma_{\mathrm{cusp}}(g)$.

1.3 Two-dimensional O(6) sigma model

Substitution of (1.6) into relation (1.3) yields a definite prediction for the energy density of the O(6) model in the regime of large particle density $\rho \gg m$. In this regime the model is known to be weakly coupled and the energy density can be computed perturbatively.

The two-dimensional O(6) sigma model is an exactly solvable quantum field theory. It is asymptotically free at short distances, while in the infrared it develops a mass gap. The massive excitations form the vector multiplet of the O(6) group and their S-matrix has a factorized form [11]. This makes it possible to calculate the mass gap m in terms of the parameter Λ. The idea is to consider the O(6) model in the presence of an external field h coupled to one of the conserved charges, say Q^{12} (see Eq. (3.1) below), and calculate the change in the free energy density in two different ways: from thermodynamical Bethe ansatz and from perturbative expansion [12].

If the external field exceeds the mass gap, $h \geq m$, a finite density of particles ρ is formed in the ground state. The corresponding ground-state energy density $\varepsilon_{\mathrm{O}(6)}(\rho)$ is given by

$$\varepsilon_{\mathrm{O}(6)} = m \int_{-B}^{B} \frac{d\theta}{2\pi}\, \chi(\theta) \cosh\theta\,, \qquad \rho = \int_{-B}^{B} \frac{d\theta}{2\pi}\, \chi(\theta)\,, \tag{1.8}$$

where the rapidity distribution satisfies an integral TBA equation

$$\chi(\theta) = \int_{-B}^{B} d\theta' K(\theta-\theta')\chi(\theta') + m\cosh\theta\,. \tag{1.9}$$

Here the kernel $K(\theta) = (\log S(\theta))'/(2\pi i)$ is given by the logarithmic derivative of the scattering matrix of the particles corresponding to the largest eigenvalue of the charge Q^{12}

$$K(\theta) = \frac{1}{4\pi^2}\left[\psi\left(1+\frac{i\theta}{2\pi}\right) + \psi\left(1-\frac{i\theta}{2\pi}\right) - \psi\left(\frac{1}{2}-\frac{i\theta}{2\pi}\right) - \psi\left(\frac{1}{2}+\frac{i\theta}{2\pi}\right) + \frac{2\pi}{\cosh\theta}\right], \tag{1.10}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ denotes the logarithmic derivative of the gamma function.

The free energy density $\mathcal{F}(h)$ can be obtained from (1.8) through the Legendre transformation, $\mathcal{F}(h) = \min_\rho[\varepsilon(\rho) - h\rho]$. Due to asymptotic freedom, for $h \gg m$ the change in the free energy density can also be calculated perturbatively with the result

$$\mathcal{F}(h) - \mathcal{F}(0) = -\frac{h^2}{\pi}\left\{\frac{1}{\alpha} - \frac{1}{2} - \frac{\alpha}{8} + O(\alpha^2)\right\}, \tag{1.11}$$

where the coupling $\alpha = \alpha(h)$ is defined in a renormalization group invariant way by

$$\frac{1}{\alpha} + \frac{1}{4}\ln\alpha = \ln\frac{h}{\Lambda_{\overline{\mathrm{MS}}}}\,, \tag{1.12}$$

with $\Lambda_{\overline{\mathrm{MS}}}$ being the Λ scale in the $\overline{\mathrm{MS}}$ scheme. The first two terms in the right-hand side of (1.11) were found in [12] while calculation of the coefficient in front of α is one of the main results of this paper.

At large h, the solution to the TBA equations (1.8) and (1.9) can be constructed using the generalized Wiener-Hopf technique [12]. The result for $\mathcal{F}(h) - \mathcal{F}(0)$ is given by series in $[\ln(h/m)]^{-1}$. The calculation is rather involved and only the first few terms have been calculated so far [12]. Matching these terms into perturbative expansion (1.11), one can establish the relation [12] between the mass gap m and the scale $\Lambda_{\overline{\rm MS}}$

$$\zeta = \ln \frac{m}{\Lambda_{\overline{\rm MS}}} = \frac{1}{4}(3\ln 2 - 1) - \ln \Gamma\left(\tfrac{5}{4}\right). \qquad (1.13)$$

Taking into account this relation, we find from (1.11) and (1.12) the following expression for the ground state energy density as a function of ρ/m

$$\varepsilon_{\rm O(6)} = \frac{\rho^2 \tilde\alpha \pi}{4} \left\{ 1 + \frac{\tilde\alpha}{2} + \frac{\tilde\alpha^2}{8} + O(\tilde\alpha^3) \right\}, \qquad (1.14)$$

where $\tilde\alpha = \tilde\alpha(\rho)$ denotes another useful coupling

$$\frac{1}{\tilde\alpha} - \frac{3}{4}\ln\tilde\alpha = \ln\frac{\rho}{m} + \ln\frac{\pi}{2} + \zeta. \qquad (1.15)$$

The relation (1.14) defines the energy density $\varepsilon_{\rm O(6)}$ as a function of the particle density ρ and the mass gap m. We can now apply (1.3) to translate it into the dependence of the scaling function $\epsilon(g,j)$ on j and m in the limit $m \ll j \ll g$.

To obtain the strong coupling expansion of $\epsilon(g,j)$ from (1.3) we also need the explicit form of the function $m = m(g)$. To leading order it is given by (1.4) while the subleading correction to m will be computed below (see Eq. (2.4)). Replacing the mass scale m in (1.14) by its explicit expression (1.4), we find that the scaling function admits the same perturbative expansion as (1.6) and provides a definite prediction for the constant q_{02}. This constant depends on subleading corrections both to the energy density $\varepsilon_{\rm O(6)}$ and to the mass scale m. We find that with these corrections taken into account the relation (1.3) leads to $q_{02} = -3\ln 2/2 + 11/4$ (see Eq. (1.7)), in agreement with the quantum string Bethe ansatz result of [30].

The paper is organized as follows. In Sect. 2 we summarize the properties of the scaling function in planar $\mathcal{N}=4$ SYM at strong coupling. We show that for $m \ll j \ll g$ the scaling function has the same form as (1.6) and obtain the expression for the constant q_{02} which involves corrections both to the mass gap m and to the energy density $\varepsilon_{\rm O(6)}$. Then, we compute subleading $O(1/g)$ correction to the mass scale (1.4). In Sect. 3, we calculate the ground-state energy density of the O(n) sigma model for large particle density ρ using perturbation theory technique. We obtain an expression for the energy density $\varepsilon_{{\rm O}(n)}$ to second order in the effective coupling constant expansion and demonstrate that it is in agreement with numerical solution of the TBA equations. Finally, we use the obtained expressions for m and $\varepsilon_{\rm O(6)}$ to compute the constant q_{02}. Section 4 contains our concluding remarks. Some technical details are presented in two Appendices.

2 Scaling function in $\mathcal{N}=4$ SYM at strong coupling

For $g \to \infty$ and $j/m =$ fixed, the scaling function $\epsilon(g,j)$ is related to the energy density of the O(6) model through relation (1.3). To compute $\epsilon(g,j)$ from (1.3) we have to accomplish two tasks: (i) to determine the energy density $\varepsilon_{\rm O(6)}$ as a function of particle density $\rho = j/2$ and mass gap m, and (ii) calculate the mass scale m from the FRS equation as a function of the coupling constant g.

The leading order expression for the mass scale is given by (1.4). The subleading correction to m can be parameterized as follows

$$m = kg^{1/4}\,\mathrm{e}^{-\pi g}\left[1+\frac{m_1}{\pi g}+O(1/g^2)\right], \tag{2.1}$$

with k defined in (1.4) and $g-$independent parameter m_1 to be determined. For the energy density $\varepsilon_{\mathrm{O}(6)}$, we make use of the results of previous studies of the TBA equations for the two-dimensional $\mathrm{O}(n)$ model [12]. For arbitrary j/m, solutions to the TBA equations (1.8) and (1.9) do not admit a simple analytical representation. However, they can be constructed in the limits $j/m \ll 1$ and $j/m \gg 1$, which correspond, respectively, to (nonperturbative) small and (perturbative) large particle density regimes:

- For $j \ll m$, the integral equations (1.8) and (1.9) can be solved by iterations leading to [12, 6, 7]

$$\epsilon(j,g)+j = m^2\left[\frac{j}{m}+\frac{\pi^2}{24}\left(\frac{j}{m}\right)^3+\ldots\right], \tag{2.2}$$

  where expansion runs in powers of j/m. This regime was extensively studied both numerically and analytically and subleading corrections to (2.2) were recently computed in [25].

- For $j \gg m$, the expression for the scaling function follows from the known perturbative result for the energy density of O(6) model [12, 6, 7]

$$\epsilon(j,g)+j = \frac{\pi j^2}{8\ln(j/m)}\left[1+\frac{3}{4}\frac{\ln(\kappa\ln(j/m))+\frac{1}{2}}{\ln(j/m)}+\frac{9}{16}\frac{\ln^2(\kappa\ln(j/m))+\epsilon_1}{\ln^2(j/m)}+\ldots\right], \tag{2.3}$$

  where $\ln\kappa = \frac{1}{2}-\frac{1}{3}\ln 2-\frac{4}{3}\ln\Gamma(\frac{3}{4})$ and the constant ϵ_1 remains unknown.

The relation (2.3) resums through renormalization group (an infinite number of) perturbative corrections in $1/g$ which are proportional to j^2 and are enhanced by powers of $\ln(j/g)$ [6, 28]. Replacing the mass scale in (2.3) by its expression (2.1) and re-expanding the right-hand side of (2.3) in powers of $1/g$, we arrive at the relation (1.6) and obtain the constant q_{02} as

$$q_{02} = \frac{9}{8}+8m_1+\frac{9}{2}\epsilon_1\,. \tag{2.4}$$

It depends on the parameters m_1 and ϵ_1 entering (2.1) and (2.3), respectively. In this section, we compute m_1 and return to ϵ_1 in section 3.

2.1 Mass scale

The dependence of the mass scale m on the coupling follows univocally from the FRS equation. As was shown in [7], it has the following form

$$m = \frac{8\sqrt{2}}{\pi^2}\,\mathrm{e}^{-\pi g} - \frac{8g}{\pi}\,\mathrm{e}^{-\pi g}\,\mathrm{Re}\left[\int_0^\infty \frac{dt\;\mathrm{e}^{i(t-\pi/4)}}{t+i\pi g}\left(\Gamma_+(t)+i\Gamma_-(t)\right)\right], \tag{2.5}$$

where $\Gamma_\pm(t)$ are real functions of t also depending on the coupling constant but independent on the scaling variable j. In addition, these functions have a definite parity, $\Gamma_\pm(-t) = \pm\Gamma_\pm(t)$, and satisfy an integral equation [24] which follows from the BES equation [18]. To save space, we do not present it here and refer an interested reader to [7, 24] for details.

It is interesting to note that the functions $\Gamma_\pm(t)$ also play a distinguished role in determination of the cusp anomalous dimension in planar $\mathcal{N}=4$ SYM theory. Namely, for arbitrary coupling $\Gamma_{\rm cusp}(g)$ can be derived from the asymptotic behavior of $\Gamma_\pm(t)$ at small t

$$\Gamma_{\rm cusp}(g) = -2g\left(\Gamma_+(0) + i\Gamma_-(0)\right). \tag{2.6}$$

This relation was used in [24] to obtain the asymptotic expansion of $\Gamma_{\rm cusp}(g)$ in powers of $1/g$.

The functions $\Gamma_\pm(t)$ were constructed in [24] in the form of Neumann series over Bessel functions

$$\begin{aligned}\Gamma_+(t) &= \sum_{k=0}^{\infty}(-1)^{k+1}J_{2k}(t)\Gamma_{2k}(g)\,, \\ \Gamma_-(t) &= \sum_{k=0}^{\infty}(-1)^{k+1}J_{2k-1}(t)\Gamma_{2k-1}(g)\,.\end{aligned} \tag{2.7}$$

Here the $g-$dependent expansion coefficients are given by

$$\Gamma_{-1}=1\,,\qquad \Gamma_k(g) = -\frac{1}{2}\Gamma_k^{(0)} + \sum_{p=1}^{\infty}\frac{1}{g^p}\left[c_p^-(g)\Gamma_k^{(2p-1)} + c_p^+(g)\Gamma_k^{(2p)}\right],\quad (k\ge 0)\,, \tag{2.8}$$

where the dependence on k is carried by the coefficients $\Gamma_k^{(p)}$ defined as

$$\Gamma_{2k}^{(p)} = \frac{\Gamma(k+p-\frac{1}{2})}{\Gamma(k+1)\Gamma(\frac{1}{2})},\qquad \Gamma_{2k-1}^{(p)} = (-1)^p\frac{\Gamma(k-\frac{1}{2})}{\Gamma(k+1-p)\Gamma(\frac{1}{2})}\,, \tag{2.9}$$

while the dependence on the coupling resides in $c_p^\pm(g)$.

The functions $\Gamma_\pm(t)$ defined in (2.7) are uniquely specified by the expansion coefficients $c_p^\pm(g)$. The latter satisfy the quantization conditions [24]

$$\begin{aligned}&\sum_{p\ge 0}(2\pi s)^p\, c_p^+(g)\frac{\Gamma(p-\frac{1}{4})}{2\Gamma(\frac{3}{4})} = \frac{\Gamma(\frac{3}{4})\Gamma\left(1-s\right)}{\Gamma\left(\frac{3}{4}-s\right)} + O(1/g)\,, \\ &\sum_{p\ge 0}(2\pi s)^p\,\left[c_p^-(g) + c_p^+(g)\left(2p-\tfrac{3}{2}\right)\left(p-\tfrac{1}{4}\right)\right]\frac{4\Gamma(p-\frac{3}{4})}{\Gamma(\frac{1}{4})} = \frac{\Gamma(\frac{1}{4})\Gamma\left(1-s\right)}{\Gamma\left(\frac{1}{4}-s\right)} + O(1/g)\,,\end{aligned} \tag{2.10}$$

with $c_0^+=-\frac{1}{2}$, $c_0^-=0$ and s being arbitrary. Comparing coefficients in front of powers of s in both sides of these relations, we can determine $c_p^\pm(g)$ in the form of asymptotic series in $1/g$. In this manner, we get

$$\begin{aligned}c_1^+(g) &= -3\,\frac{\ln 2}{\pi} + \frac{1}{2} + O(1/g)\,, \\ c_1^-(g) &= \frac{3}{4}\,\frac{\ln 2}{\pi} - \frac{1}{4} + O(1/g)\,,\quad \ldots\end{aligned} \tag{2.11}$$

Substituting these relations into (2.7) and (2.8) and performing summation over k in the right-hand side of (2.8) we obtain after some algebra

$$\Gamma_+(it) + i\Gamma_-(it) = -V_0(t) - (4\pi g)^{-1}\left[\left(\frac{\pi}{2} - 3\ln 2\right)tV_0(t) - \frac{3\ln 2}{2}V_1(t)\right] + \mathcal{O}\left(1/g^2\right), \tag{2.12}$$

where the notation was introduced for the functions

$$V_0(t) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} du\, \mathrm{e}^{tu} \left(\frac{1+u}{1-u}\right)^{1/4}, \qquad (2.13)$$
$$V_1(t) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} du\, \mathrm{e}^{tu} \left(\frac{1+u}{1-u}\right)^{1/4} \frac{1}{1+u}.$$

The first term in the right-hand side of (2.12) coincides with the leading-order solution found in [21]. Moreover, taking $t = 0$ in (2.12) we verify using (2.6) that

$$\Gamma_{\rm cusp}(g)/(2g) = -\Gamma_+(0) - i\Gamma_-(0) = 1 - \frac{3\ln 2}{4\pi g} + \dots, \qquad (2.14)$$

in agreement with the known strong coupling expansion of the cusp anomalous dimension.

2.2 Correction to the mass gap

Let us now apply (2.5) and (2.7) to compute the mass scale at strong coupling. We observe that the $t-$integral in the right-hand side of (2.5) receives a dominant contribution from the region $t \sim g$. Trying to apply (2.12) we recognize that the expansion (2.12) is not well-defined in this region because, due to the presence of t inside square brackets, expansion parameter is $t/(4\pi g) = O(g^0)$. Thus, in order to compute the mass scale from (2.5), we have to resum the whole series (2.12) in the double scaling limit $t, g \to \infty$ with t/g = fixed. Fortunately, this particular limit was already studied in [24].

In the double scaling limit, it is convenient to change the integration variable in (2.5) as $t \to 4\pi git$ and expand the function $\Gamma_\pm(4\pi git)$ into series in $1/g$ with $t = O(g^0)$. We find from (2.12) that the expansion has the following form

$$\Gamma_+(4\pi git) + i\Gamma_-(4\pi git) = f_0(t)V_0(4\pi gt) + f_1(t)V_1(4\pi gt), \qquad (2.15)$$

with $f_0(t) = -1 - \left(\frac{\pi}{2} - 3\ln 2\right)t + O(t^2, 1/g)$ and $f_1(t) = O(1/g)$. To determine the functions $f_0(t)$ and $f_1(t)$, we solve the quantization conditions (2.10) and, then, use the obtained expressions for $c_p^\pm(g)$ to resum the series in (2.8). In this way, we obtain that $f_0(t)$ and $f_1(t)$ are given by a linear combination of the ratio of Euler gamma-functions (see Eq. (A.4) in Appendix A). Their substitution into (2.15) yields the expression for functions $\Gamma_\pm(4\pi git)$ in the double scaling limit [26]

$$\begin{aligned}\Gamma_+(4\pi git) + i\Gamma_-(4\pi git) = &- V_0(4\pi gt)\frac{\Gamma(\frac34)\Gamma(1-t)}{\Gamma(\frac34 - t)} \qquad (2.16)\\ &+ \frac{V_1(4\pi gt)}{4\pi g}\left[\frac{\Gamma(\frac14)\Gamma(1+t)}{4t\,\Gamma(\frac14+t)} - \frac{\Gamma(\frac34)\Gamma(1-t)}{4t\,\Gamma(\frac34 - t)}\right]\\ &+ \frac{V_0(4\pi gt)}{4\pi g}\left[\left(\frac{3\ln 2}{4} + \frac{1}{8t}\right)\frac{\Gamma(\frac34)\Gamma(1-t)}{\Gamma(\frac34 - t)} - \frac{\Gamma(\frac14)\Gamma(1+t)}{8t\,\Gamma(\frac14+t)}\right]\Bigg\} + \dots,\end{aligned}$$

where ellipses denote subleading terms suppressed by powers of $1/g$ and the functions $V_0(4\pi gt)$ and $V_1(4\pi gt)$ are defined in (2.13). The relation (2.16) is consistent with the expansion (2.12) and it can be used to compute the $1/g$ correction to the mass gap (2.5).

We are now ready to compute the mass scale (2.5). To this end, we substitute (2.16) into (2.5) and work out the asymptotic expansion of the $t-$integral at large g. For the first term in the right-hand side of (2.16) this calculation was already performed in [7]. It leads to the expression for m which coincides with the mass gap (1.4) found from the string theory considerations [6]. Taking into account the remaining terms in the right-hand side of (2.16) we should be able to compute the subleading correction to m. Calculation goes along the same lines as in [7] and it leads to (see Appendix A for more details)

$$m = kg^{1/4}\,\mathrm{e}^{-\pi g}\left[1 + \frac{3-6\ln 2}{32\pi g} + O(1/g^2)\right], \tag{2.17}$$

with $k = 2^{3/4}\pi^{1/4}/\Gamma(5/4)$. Comparing this relation with (2.1) we conclude that

$$m_1 = \frac{3}{32} - \frac{3}{16}\ln 2\,. \tag{2.18}$$

This result is in an agreement with the numerical value found in the last reference in [25].

The coefficient in front of $(32\pi g)^{-1}$ in (2.17) is given by the sum of two terms of different transcendentality. We observe that $\sim \ln 2$ term can be absorbed into redefinition of the coupling constant

$$g' = g - \frac{3\ln 2}{4\pi}\,, \tag{2.19}$$

so that the mass gap (2.17) in terms of the shifted coupling g' looks as

$$m = \frac{(\pi g')^{1/4}\,\mathrm{e}^{-\pi g'}}{\Gamma(5/4)}\left[1 + \frac{3}{32\pi g'} + O(1/g'^2)\right]. \tag{2.20}$$

It is interesting to notice that similar simplification also occurs for the cusp anomalous dimension $\Gamma_{\rm cusp}(g)$ at strong coupling. As was found in [24], the expansion coefficients in the strong coupling expansion of $\Gamma_{\rm cusp}(g)$ also involve terms $\sim \ln 2$ but they disappear after one re-expands the series in $1/g'$. This suggests that the expansion parameter at strong coupling is g' rather than g.

2.3 Induced renormalization scheme

As was already mentioned, the mass scale m emerges in the AdS/CFT correspondence through dimensional transmutation mechanism in an effective two-dimensional theory describing dynamics of massless excitations in the $\mathrm{AdS}_5\times\mathrm{S}^5$ sigma model. As a result, the dependence of the mass scale of the coupling constant is dictated by the renormalization group.

The coupling constant in the effective theory depends on the scale and is related to the coupling of the (conformal invariant) $\mathrm{AdS}_5\times\mathrm{S}^5$ sigma model as $\bar g^2(\mu) = 1/(2g)$ with the scale $\mu \sim 1$ defined by masses of massive excitations [6, 28]. The coupling $\bar g(\mu)$ satisfies the Gell-Man–Low equation

$$\mu\frac{d\bar g}{d\mu} = \beta(\bar g) = -\beta_0\bar g^3 - \beta_1\bar g^5 - \beta_2\bar g^7 + O(\bar g^9)\,, \tag{2.21}$$

and it leads to the following expression for a renormalization group invariant scale

$$\Lambda = \mu\,\mathrm{e}^{-\int^{\bar g}\frac{d\bar g}{\beta(\bar g)}} = \mu\,\mathrm{e}^{-\frac{1}{2\beta_0\bar g^2}}\,\bar g^{-\beta_1/\beta_0^2}\left[1 + \frac{1}{2\beta_0}\left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0}\right)\bar g^2 + O(\bar g^4)\right]. \tag{2.22}$$

Then, the relation (1.3) between the scaling function and O(6) sigma model implies that $\beta(\bar{g})$ in (2.21) should coincide with the beta-function of this model. In bosonic two-dimensional O(6) sigma model the beta-function coefficients are given by [31]

$$\beta_0 = \frac{1}{\pi}\,, \qquad\qquad \beta_1 = \frac{1}{2\pi^2}\,. \tag{2.23}$$

Notice that the beta function (2.21) depends on the renormalization scheme starting from $O(\bar{g}^7)$ term. The same is true for the scale Λ while the mass scale m should be scheme independent. The two scales are related to each other as

$$m = c\,\Lambda\,, \tag{2.24}$$

where the coupling independent factor c is needed to restore the scheme independence of m. In the special case of the $\overline{\text{MS}}$ scheme, this relation takes the form (1.13) with $c_{\overline{\text{MS}}} = \mathrm{e}^{\zeta}$.

Replacing the beta-function coefficients in (2.22) by their actual values (2.23) we obtain from (2.24)

$$m = c\,\mu\,\mathrm{e}^{-\frac{\pi}{2\bar{g}^2}}\,\bar{g}^{-1/2}\left[1 + \left(\frac{1}{8\pi} - \frac{\pi^2}{2}\beta_2\right)\bar{g}^2 + O(\bar{g}^4)\right]. \tag{2.25}$$

Let us now compare this relation with the expression for the mass scale (2.17) obtained from exact solution of the FRS equation. We find that the two expressions indeed coincide upon identification

$$\bar{g}^2(\mu) = \frac{1}{2g}\,, \qquad c^{(\text{FRS})}\,\mu = \frac{2^{1/2}\pi^{1/4}}{\Gamma(5/4)}\,, \qquad \beta_2^{(\text{FRS})} = \frac{1}{8\pi^3}\left(6\ln 2 - 1\right). \tag{2.26}$$

Here we introduced the subscript (FRS) to indicate that these expressions are valid in a particular renormalization scheme dictated by the FRS equation.

3 Energy density in the two-dimensional O(n) sigma model

The aim of this section is to calculate the ground-state energy density of the O(n) sigma model for large particle density ρ. The ground-state energy density $\varepsilon_{\mathrm{O}(n)}$ can be obtained either from the solution of the integral TBA equations, Eqs. (1.9) and (1.8), or from perturbation theory. To compute the constant (2.4) and, then, to make a comparison with the results of Refs. [28] and [30], one needs the second subleading correction to $\varepsilon_{\mathrm{O}(6)}$ in the large ρ limit analogous to ϵ_1 in (2.3). Since the techniques on the TBA side are not developed enough to perform the expansion at such depth, we determine this correction using standard perturbation theory.

3.1 Perturbative calculation of the free energy density

The fundamental fields of the O(n) nonlinear sigma model are $\boldsymbol{S}(x) = (S^1, \ldots, S^n)$ subject to the constraint $\sum_1^n S^j S^j = 1$. The theory has global O(n) symmetry and the corresponding conserved charges can be written as

$$Q^{ij} = \int J_0^{ij}\,dx_1\,, \qquad J_\mu^{ij} = S^i\partial_\mu S^j - S^j\partial_\mu S^i\,. \tag{3.1}$$

We couple an external field to the conserved charge Q^{12} and define the theory by its Euclidean two-dimensional Lagrangian:

$$\mathcal{L}(x) = \frac{1}{2\lambda^2}\left[\partial_\mu \boldsymbol{S}\,\partial_\mu \boldsymbol{S} + 2ih(S^1\partial_0 S^2 - S^2\partial_0 S^1) + h^2\left\{(S^3)^2 + \cdots + (S^n)^2 - 1\right\} - 2\omega^2 S^1\right], \quad (3.2)$$

where λ is the bare coupling constant and $h-$dependent terms are chosen in such a way that they modify the Hamiltonian of the model by term $(-hQ^{12})$.

In order to avoid infrared divergences we introduced in the right-hand side of (3.2) an extra term with regulator ω which is going to be put to zero at the end of the calculation. This extra term fixes the classical ground-state to be

$$S^1 = 1\,, \qquad S^2 = S^3 = \cdots = S^n = 0\,. \quad (3.3)$$

We parameterize the small fluctuations around this ground-state by exploiting the remaining symmetries:

$$S^1 = \sqrt{1-\lambda^2(y^2+\boldsymbol{z}^2)}\,, \qquad S^2 = \lambda y, \quad S^3 = \lambda z^1, \quad \ldots, \quad S^n = \lambda z^{n-2}\,, \quad (3.4)$$

where the fields $\boldsymbol{z} = (z^1, \ldots, z^{n-2})$ form the vector representation of the unbroken $\mathrm{O}(n-2)$. We substitute (3.4) into (3.2) and expand the Lagrangian to second order in the coupling λ to get

$$\mathcal{L}(x) = \lambda^{-2}\mathcal{L}_{-2} + \mathcal{L}_0 + \lambda\mathcal{L}_1 + \lambda^2\mathcal{L}_2 + O(\lambda^3)\,, \quad (3.5)$$

where the various terms depend on the parameter $M^2 = h^2 + \omega^2$ and look as follows

$$\begin{aligned}
\mathcal{L}_{-2} &= -\frac{1}{2}(M^2+\omega^2)\,, \qquad (3.6)\\
\mathcal{L}_0 &= \frac{1}{2}\left[\partial_\mu y\partial_\mu y + \partial_\mu \boldsymbol{z}\partial_\mu \boldsymbol{z} + \omega^2 y^2 + M^2 \boldsymbol{z}^2\right],\\
\mathcal{L}_1 &= -ih(y^2+\boldsymbol{z}^2)\partial_0 y\,,\\
\mathcal{L}_2 &= \frac{1}{2}(y\partial_\mu y + \boldsymbol{z}\partial_\mu \boldsymbol{z})(y\partial_\mu y + \bar{z}\partial_\mu \boldsymbol{z}) + \frac{\omega^2}{8}(y^2+\boldsymbol{z}^2)^2\,.
\end{aligned}$$

Here $\mathcal{L}_{-2}$ is just a constant, $\mathcal{L}_0$ defines the kinetic term for the two-dimensional fields $y(x)$ and $\boldsymbol{z}(x)$ while $\mathcal{L}_1$ and $\mathcal{L}_2$ define, correspondingly, cubic and quartic interaction vertices. Notice that the Lorentz covariance of $\mathcal{L}_1$ is broken by the external field.

Our goal is to calculate the change in the free energy density $\mathcal{F}(h) - \mathcal{F}(0)$ relative to its value for $h = 0$. The free energy density is defined as

$$\mathrm{e}^{-V\mathcal{F}(h)} = \int \mathcal{D}y\mathcal{D}\boldsymbol{z}\,\mathrm{e}^{-\int d^Dx\,\mathcal{L}(x)}\,, \qquad D = 2-\epsilon\,. \quad (3.7)$$

It is ultraviolet divergent and we used dimensional regularization to define it properly. Substitution of (3.5) into (3.7) yields the perturbative expansion

$$\begin{aligned}
\mathrm{e}^{-V\mathcal{F}(h)} &= \mathrm{e}^{\frac{M^2+m^2}{2\lambda^2}V}\int \mathcal{D}y\mathcal{D}\boldsymbol{z}\,\mathrm{e}^{-\int d^Dx\,\mathcal{L}_0(x)} \qquad (3.8)\\
&\times\left[1 - \lambda^2\int d^Dx\,\mathcal{L}_2(x) + \frac{\lambda^2}{2}\int d^Dx\,\mathcal{L}_1(x)\int d^Dx'\mathcal{L}_1(x') + O(\lambda^4)\right].
\end{aligned}$$

Here the term linear in $\mathcal{L}_1$ is absent since it only involves odd powers of $y-$field and, therefore, vanishes upon integration. Taking the logarithm of both sides of (3.8), dividing by volume $V = \int d^D x$ and subtracting $\mathcal{F}(0)$ we get

$$\mathcal{F}(h) - \mathcal{F}(0) = \frac{1}{\lambda^2}\mathcal{F}^{(-1)}(h) + \mathcal{F}^{(0)}(h) + \lambda^2 \mathcal{F}^{(1)}(h) + O(\lambda^4)\,. \tag{3.9}$$

We calculate each term separately for a finite ω and put $\omega \to 0$ at the end of the calculation. We relegate the details of the calculations to Appendix B and present here only the results.

The first term in the right-hand side of (3.9) describes the classical contribution to the free energy

$$\mathcal{F}^{(-1)}(h) = -\frac{h^2}{2}\,. \tag{3.10}$$

The next order term, $\mathcal{F}^{(0)}(h)$, sums up the quadratic fluctuations of $y-$ and $\boldsymbol{z}-$fields and it is related to the determinant of the kinetic operators in $\mathcal{L}_0$. Using dimensional regularization it can be written as

$$\mathcal{F}^{(0)}(h) = \frac{n-2}{4\pi} h^{2-\epsilon}\left\{\frac{1}{\epsilon} + \frac{\gamma}{2} + \frac{1}{2}\right\}, \qquad \gamma = \Gamma'(1) + \ln(4\pi)\,. \tag{3.11}$$

Finally, going through calculation of the the third term in the right-hand side of (3.9) we find

$$\mathcal{F}^{(1)}(h) = \frac{n-2}{16\pi^2} h^{2-2\epsilon}\left\{\frac{1}{\epsilon} + \gamma + \frac{1}{2}\right\}. \tag{3.12}$$

At this point, we combine together three terms and obtain the following expression for the free energy density

$$\mathcal{F}(h) - \mathcal{F}(0) = -\frac{h^2}{2\lambda^2} + \frac{n-2}{4\pi} h^{2-\epsilon}\left\{\frac{1}{\epsilon} + \frac{\gamma}{2} + \frac{1}{2}\right\} + \lambda^2 \frac{n-2}{16\pi^2} h^{2-2\epsilon}\left\{\frac{1}{\epsilon} + \gamma + \frac{1}{2}\right\} + O(\lambda^4)\,. \tag{3.13}$$

The first two terms in this expansion were already calculated in [12]. The result for the $O(\lambda^2)$ term is new and it is needed to make the comparison with the results of Refs. [28] and [30].

3.2 Renormalization of the free energy density

Ultraviolet divergences in the free energy (3.13) can be removed in the standard way by expressing the free energy in term of the renormalized coupling and using the renormalization group to improve the result. In the $\overline{\text{MS}}$ scheme, the relation between bare coupling λ and renormalized coupling $\tilde{g}(\mu)$ reads

$$\lambda^2 \to \left(\mu\, \mathrm{e}^{\gamma/2}\right)^{\epsilon} Z_1 \tilde{g}^2\,, \qquad Z_1 = 1 - \frac{2\beta_0 \tilde{g}^2}{\epsilon} - \frac{\beta_1 \tilde{g}^4}{\epsilon} + \frac{4\beta_0^2 \tilde{g}^4}{\epsilon^2} + \dots\,, \tag{3.14}$$

where γ is defined in (3.11) and $\beta_{0,1,2}$ are the coefficients of the $\beta-$function of the O(n) model up to three loops

$$\mu \frac{d\tilde{g}}{d\mu} = \beta(\tilde{g}) = -\beta_0 \tilde{g}^3 - \beta_1 \tilde{g}^5 - \beta_2 \tilde{g}^7 + \dots \tag{3.15}$$

In the $\overline{\text{MS}}$ scheme they have been determined in [31] to be

$$\beta_0 = \frac{n-2}{4\pi}\,, \qquad \beta_1 = \frac{n-2}{8\pi^2}\,, \qquad \beta_2^{(\overline{\text{MS}})} = \frac{(n+2)(n-2)}{64\pi^3}\,, \tag{3.16}$$

where we introduced the superscript to indicate that β_2 is scheme-dependent.

Note that the coupling $\tilde{g}$ introduced here is analogous to the coupling $\bar{g}$ introduced in Sect. 2.3. However the important difference between the two couplings is that they are defined in two different schemes. Indeed, it is straightforward to verify that for $n = 6$ the beta-functions (2.23) and (3.16) coincide up to two loops, but they differ starting from three loops, $\beta_2^{(\text{FRS})} \neq \beta_2^{(\overline{\text{MS}})}$. Still, the two couplings are related to each other through a finite renormalization.

With the relation (3.14) taken into account, the renormalized free energy density reads

$$\mathcal{F}(h) - \mathcal{F}(0) = -\frac{h^2}{2}\left\{\frac{1}{\tilde{g}^2} - 2\beta_0\left(\ln\frac{\mu}{h} + \frac{1}{2}\right) - 2\beta_1\tilde{g}^2\left(\ln\frac{\mu}{h} + \frac{1}{4}\right) + O(\tilde{g}^4)\right\}\,. \tag{3.17}$$

We verify with a help of (3.15) that the right-hand side of this relation does not depend on the renormalization scale μ. This suggests to express $\mathcal{F}(h) - \mathcal{F}(0)$ in a renormalization group invariant way.

It is important to keep in mind that, contrary to the free energy density, the coupling $\tilde{g}(\mu)$ is not a physical quantity. We can explore this fact to define a new coupling constant to our best convenience. The running of the coupling is determined by the Gell-Mann–Low equation (3.15) and it depends on the scale $\Lambda_{\overline{\text{MS}}}$ given by a general expression (2.22) in the $\overline{\text{MS}}$ scheme. Using this scale we define a new universal coupling $\alpha(h)$ as

$$\frac{1}{\alpha} + \xi \ln\alpha = \ln\frac{h}{\Lambda_{\overline{\text{MS}}}}\,, \qquad \xi = \frac{\beta_1}{2\beta_0^2} = \frac{1}{n-2}\,. \tag{3.18}$$

Differentiating both sides of (3.18) with respect to h we find the coupling α defines the scheme in which beta-function is given to all loops by

$$h\frac{d\alpha}{dh} = \beta(\alpha) = -\frac{\alpha^2}{1-\xi\alpha}\,. \tag{3.19}$$

It has the advantage that any other coupling constant depends only polynomially on α. In particular, for $\mu = h$ we find from (3.15) and (3.18)

$$\tilde{g}^2(h) = \frac{1}{2\beta_0}\left(\alpha + \frac{\xi}{4}\alpha^3 + O(\alpha^4)\right)\,. \tag{3.20}$$

Then, perturbative expansion of the free energy density (3.17) in α takes the form

$$\mathcal{F}(h) - \mathcal{F}(0) = -\beta_0 h^2\left\{\frac{1}{\alpha} - \frac{1}{2} - \frac{\xi\alpha}{2} + O(\alpha^2)\right\}\,. \tag{3.21}$$

Here the first two terms reproduce the result of [12] and the third one is the sought next order correction.

Let us now determine perturbative expansion of the energy density $\varepsilon(\rho)$. It can be obtained from the free energy density (3.21) through Legendre transformation

$$\varepsilon(\rho) = \mathcal{F}(h) - \mathcal{F}(0) + \rho h\,, \qquad \rho = -\mathcal{F}'(h)\,. \tag{3.22}$$

Using (3.18) and (3.21) we find explicitly

$$\varepsilon = \frac{\rho^2 \alpha(h)}{4\beta_0} \left[1 + \frac{\alpha(h)}{2} + \frac{\xi \alpha^2(h)}{2} + O(\alpha^2) \right], \qquad (3.23)$$
$$\rho = 2\beta_0 h \left[\frac{1}{\alpha(h)} + O(\alpha^2) \right].$$

These relations define a parametric dependence of the energy density ε on the particle density ρ. To express ε entirely as a function of ρ we introduce yet another coupling $\tilde{\alpha}(\rho)$ defined by

$$\frac{1}{\tilde{\alpha}} + (\xi - 1) \ln \tilde{\alpha} = \ln \frac{\rho}{2\beta_0 \Lambda_{\overline{MS}}}. \qquad (3.24)$$

Replacing ρ by its expression (3.23) and making use of (3.18), we establish the relation between the two couplings

$$\frac{1}{\tilde{\alpha}(\rho)} + (\xi - 1) \ln \tilde{\alpha}(\rho) = \frac{1}{\alpha(h)} + (\xi - 1) \ln \alpha(h) + O(\alpha^3), \qquad (3.25)$$

leading to $\alpha(h) = \tilde{\alpha}(\rho) + O(\tilde{\alpha}^4)$.

As a result, we obtain from (3.23) the energy density in the O(n) model as

$$\varepsilon(\rho) = \rho^2 \pi \xi \left\{ \tilde{\alpha} + \frac{\tilde{\alpha}^2}{2} + \frac{\xi \tilde{\alpha}^3}{2} + O(\tilde{\alpha}^4) \right\}. \qquad (3.26)$$

According to (3.24), the coupling $\tilde{\alpha}$ depends on the scale $\Lambda_{\overline{\text{MS}}}$. To make a comparison with the string theory calculation we need its expression in terms of the physical mass m. To this end, we take into account the known relation between the two scales [12]

$$\zeta = \ln \frac{m}{\Lambda_{\overline{\text{MS}}}} = (3 \ln 2 - 1)\xi - \ln \Gamma(1 + \xi), \qquad (3.27)$$

and express the coupling (3.24) in terms of ρ/m as

$$\frac{1}{\tilde{\alpha}} + (\xi - 1) \ln \tilde{\alpha} = \ln \frac{2\rho}{m} + A, \qquad A = \zeta + \ln(\pi\xi) \qquad (3.28)$$

with $\xi = 1/(n-2)$. Being combined together, the relations (3.26) and (3.28) define the first few terms of the perturbative expansion of the energy density in the O(n) model in the perturbative regime $\rho \gg m$.

3.3 Numerical analysis of the TBA equation

As shown in the second paper in [12], the same function $\varepsilon(\rho)$ can be obtained from the solution of the integral TBA equation (1.9) in a parametric form given in (1.8). Using the generalized Wiener-Hopf technique one can transform equation (1.9) to a form more suitable both for finding the large ρ asymptotic expansion of the energy density and for numerical calculation. Employing this technique, Hasenfratz, Maggiore and Niedermayer [12] calculated the first two terms in the large ρ expansion of $\varepsilon(\rho)$ and computed the ratio $m/\Lambda_{\overline{\text{MS}}}$.

Since for the calculation of the next order term we have to leave the beaten paths we decided to check the result of the perturbation theory numerically for the O(6) model. The precision measurement would require to perform several integral transformation, but a rough estimate can be obtained directly form the original TBA equations (1.8) and (1.9) as follows: First the parameter B is fixed as an integer in the range $B = 5, \ldots, 14$. On the interval $[-B, B]$ the pseudo-energy $\chi(\theta)$ is discretized on $2^{18} \div 2^{24}$ points. Then the integral equation (1.9) is solved by iteration. (Technically, we used the fast Fourier transformation routine of the numerical programming language octave to perform the convolution). From the solution, $\chi(\theta, B)$, we calculated the ratios $\rho(B)/m$ and $\varepsilon(B)/m^2$ with a help of (1.8). Our relative precision for these quantities was as good as 10^{-5}. Then we calculated the coupling $\tilde{\alpha}(\rho)$ by numerically solving the equation (3.28). The results are displayed in Table 1.

B	5	6	7	8	9
$\tilde{\alpha}$	0.23355(3)	0.19097(5)	0.16135(3)	0.13958(8)	0.12293(8)
ρ/m	94.8911(7)	286.651(8)	850.661(7)	2492.50(2)	7233.93(2)
ε/m^2	1856.75(4)	13560.90(5)	99410.4(0)	730337.7(8)	5373044.(2)
B	10	11	12	13	14
$\tilde{\alpha}$	0.10979(9)	0.099173(3)	0.090405(5)	0.083049(9)	0.076792(3)
ρ/m	$2.08399(8) \cdot 10^4$	$5.96823(1) \cdot 10^4$	$1.70101(3) \cdot 10^5$	$4.82856(7) \cdot 10^5$	$1.36601(2) \cdot 10^6$
ε/m^2	$3.95663(6) \cdot 10^7$	$2.91555(6) \cdot 10^8$	$2.14945(9) \cdot 10^9$	$1.58524(9) \cdot 10^{10}$	$1.16947(3) \cdot 10^{11}$

Table 1: Numerical result for the energy and particle density in the O(6) sigma model as function of parameter B. Unreliable digits are displayed in parenthesis.

Finally we fitted the ratio $\varepsilon/(\rho^2\tilde{\alpha})$ to the expression

$$\frac{\varepsilon}{\rho^2\tilde{\alpha}} = \frac{\pi}{4}\left(1 + \frac{\tilde{\alpha}}{2} + \frac{\tilde{\alpha}^2}{8} + O(\tilde{\alpha}^3)\right), \tag{3.29}$$

which follows from (3.26) in the case of the O(6) model. Feeding in the known first two coefficients we obtained 0.12(4) for the coefficient of the interesting $O(\tilde{\alpha}^2)$ term in the right-hand side of (3.29). This is in good agreement with the predicted value $\frac{1}{8}$, recalling that here we are dealing with the coefficient of a second subleading correction.

To visualize the result we collect the data on Figure 1, where we plot $\varepsilon/(\rho^2\tilde{\alpha})$ against the coupling $\tilde{\alpha}$. Clearly the sum of all the three calculated perturbative contributions approaches nicely the numerical solution giving a convincing confirmation of both. We obtained similar result for the O(3) model as well, thus confirming our perturbative calculations numerically.

3.4 Large j expansion of the energy density

To make a decisive comparison with the results of Refs. [28] and [30], we have to examine the behavior of the energy density $\varepsilon(\rho)$ for large particle density $\rho \gg m$, or equivalently $j \gg m$ with the scaling variable $j = 2\rho$ defined in (1.3).

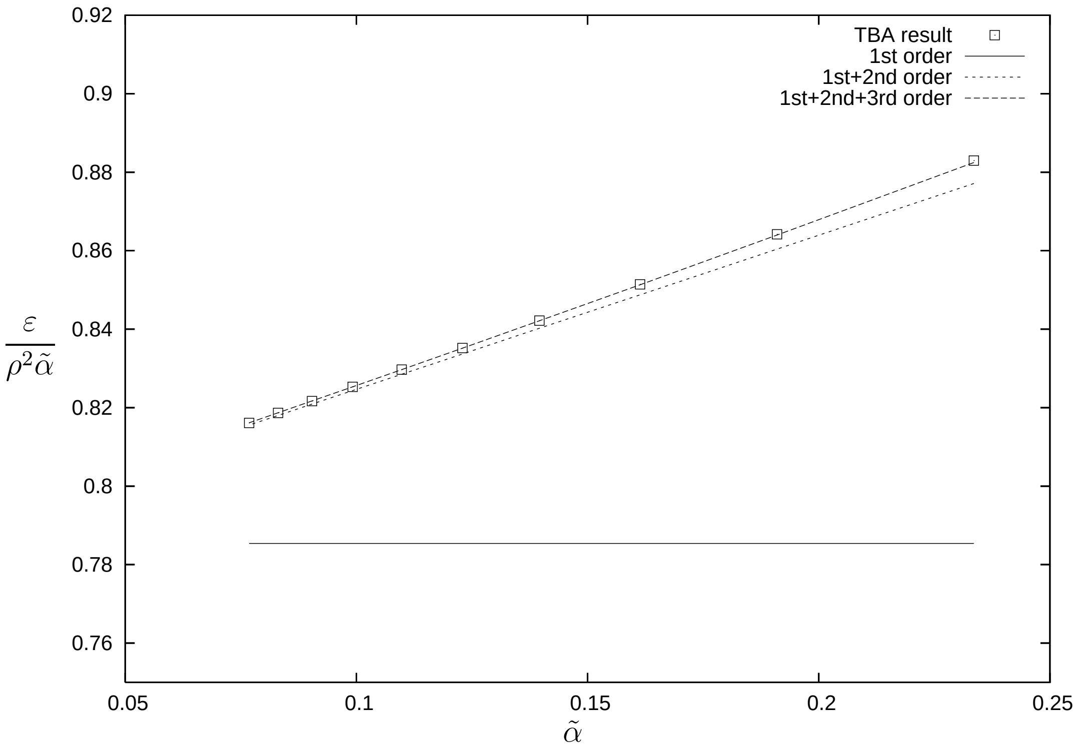

Figure 1: $\varepsilon/(\rho^2\tilde{\alpha})$ is plotted against $\tilde{\alpha}$. Boxes represent the numerical TBA data with invisible numerical errors. The lines correspond to perturbative corrections as marked on the figure.

In this region, the coupling constant (3.28) is small and it has the form

$$\frac{1}{\tilde{\alpha}} = \ln\frac{j}{m} + a_1 \ln\ln\frac{j}{m} + a_2 + a_3 \frac{\ln\ln\frac{j}{m}}{\ln\frac{j}{m}} + \frac{a_4}{\ln\frac{j}{m}} + \dots \,. \tag{3.30}$$

The coefficients a_i can be determined recursively from (3.28) leading to

$$a_1 = \xi - 1, \qquad a_2 = A, \qquad a_3 = (\xi-1)^2, \qquad a_4 = (\xi - 1)A\,, \tag{3.31}$$

with the constants ξ and A defined in (3.18) and (3.28), respectively. We substitute the relation (3.30) into (3.26) and obtain, after some algebra, the following expression for the energy density of the O(n) model

$$\begin{aligned} \varepsilon_{\mathrm{O}(n)} = \frac{j^2\pi\xi}{4\ln\frac{j}{m}} \Bigg\{ 1 &+ \frac{(1-\xi)}{\ln\frac{j}{m}}\left[\ln\left(\kappa\ln\frac{j}{m}\right) + \frac{1}{2}\right] \\ &+ \frac{(1-\xi)^2}{\ln^2\frac{j}{m}}\left[\ln^2\left(\kappa\ln\frac{j}{m}\right) + \frac{\frac{\xi}{2}\left(1-\frac{\xi}{2}\right)}{(1-\xi)^2}\right] + \dots \Bigg\}, \end{aligned} \tag{3.32}$$

where the notation was introduced for

$$\ln\kappa = \frac{\frac{\xi}{2} - A}{1-\xi} = \frac{\left(\frac{3}{2} - 3\ln 2\right)\xi + \ln\left(\Gamma(\xi)/\pi\right)}{1-\xi}\,. \tag{3.33}$$

We recall that the parameter $j = 2\rho$ defines the density of particles with mass m and the relation (3.32) only holds for $j \gg m$.

We are now ready to perform a comparison of the energy density of the O(6) model and the scaling function (2.3) in the AdS/CFT. For $n = 6$ the constants (3.33) take the following values

$$\xi = \frac{1}{4}\,, \qquad\qquad \ln\kappa = \frac{1}{2} - \frac{1}{3}\ln 2 - \frac{4}{3}\ln\Gamma\left(\tfrac{3}{4}\right). \tag{3.34}$$

Taken into account these relations we find that, in agreement with (1.3), the expression for $2\varepsilon_{\mathrm{O}(6)}$ indeed coincides with the two-loop result for the scaling function (2.3) with

$$\epsilon_1 = \frac{\frac{\xi}{2}(1-\frac{\xi}{2})}{(1-\xi)^2}\bigg|_{n=6} = \frac{7}{36}\,. \tag{3.35}$$

Finally, we substitute the relations (2.18) and (3.35) into (2.4) and compute the constant q_{02} as

$$q_{02} = \frac{9}{8} + \frac{3}{4} - \frac{3}{2}\ln 2 + \frac{9}{2}\cdot\frac{7}{36} = \frac{11}{4} - \frac{3}{2}\ln 2\,. \tag{3.36}$$

Comparing this relation with (1.7) we conclude that our result for q_{02} is in agreement with the result obtained from the quantum string Bethe ansatz [30].

4 Conclusions

In this paper, we applied methods of integrable models previously developed for the two-dimensional O(n) sigma-model to study the scaling function in the AdS/CFT. The starting point of our analysis were the relations (1.3) and (1.4) which relate the energy density of the O(6) model to the scaling function and identify the mass gap of the O(6) model m with a new dynamical scale in the AdS/CFT.

These relations are extremely nontrivial given the fact that the scale m has a different origin in the models under consideration. The O(6) model is asymptotically free at short distances and the mass scale arises due to a nontrivial dynamics in the infrared. At the same time, $\mathcal{N} = 4$ SYM and string sigma-model on $\mathrm{AdS}_5 \times \mathrm{S}^5$ are conformal invariant theories and, therefore, they do not generate any scale. In the string theory, the scaling function describes quantum corrections to the folded string spinning on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and it is the underlying classical configuration that introduces the mass scales. In a similar manner, in $\mathcal{N} = 4$ SYM theory the scaling function follows from the analysis of the Bethe ansatz equations in the limit (1.1). In this limit, the Bethe roots condense on the real axis and their distribution depends on the parameters (1.1) fixed by the quantum numbers of Wilson operators. It is therefore remarkable that our calculation of the mass scale in $\mathcal{N} = 4$ SYM theory reproduces the known result for the mass gap in the O(6) model in the special renormalization scheme dictated by the FRS equation.

For $g \to \infty$ and $j/m =$ fixed, the scaling function can be found exactly by solving the TBA equations for the energy density of the O(6) model. Finding solution to these equations in the perturbative regime $j \gg m$ beyond the leading order proves to be a difficult task. We demonstrated that the problem can be circumvented by employing methods of standard perturbation theory to calculating the free energy of the O(n) model in the presence of an external field. In this way, we computed the two-loop correction to the energy density of the O(n) model and verified that it agrees with the numerical solution to the TBA equation. Then, we combined this correction (for $n = 6$) with the two-loop correction to the mass scale m and calculated the scaling function.

Comparing the obtained expression with two different predictions for the scaling function coming from two-loop stringy corrections to the folded spinning string solution [28] and from thermodynamical limit of quantum string Bethe ansatz equations [15, 29, 30], we found an agreement with the latter one. The agreement should not be surprising since the FRS equation was originally obtained from the all-loop Bethe ansatz for the dilatation operator in $\mathcal{N} = 4$ SYM theory in an appropriate scaling limit.

The question remains however what is the reason for a disagreement between our result for the scaling function and the explicit two-loop stringy calculation. Notice that the difference only amounts to a constant term (see Eqs. (1.6) and (1.7)) while logarithmically enhanced terms coincide. This implies that the two-loop stringy result is consistent with the relation (1.3) between the scaling function and the energy density of the O(6) model but it leads to the expression for the two-loop correction to the mass scale (2.1) which is different from (2.18),

$$m_1^{\rm str} = -\frac{1}{32} - \frac{3}{16}\ln 2 - \frac{1}{4}\,\mathrm{K}\,. \tag{4.1}$$

This suggests that either the all-loop Bethe ansatz does not predict correctly the mass scale, or the two-loop stringy calculation needs to be revisited. At present stage we can not discriminate between these two scenarios and the question requires further investigation. [5]

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Appendix A: Calculation of the mass scale

Let us now compute the mass gap (2.5). At large g the integral in (2.5) receives a dominant contribution from $t \sim g$. In order to evaluate (2.5) it is convenient to change the integration variable as $t \to 4\pi g i t$. Then, we get from (2.5)

$$m = \frac{8\sqrt{2}}{\pi^2}\,\mathrm{e}^{-\pi g} + \Delta m\,, \tag{A.1}$$

[5]A potential difficulty in comparing the two predictions is that the string computation [28] was performed in a scheme in which two-loop beta-function coefficient is zero. This scheme is related to the $\overline{\rm MS}$ scheme by a singular coupling redefinition and it was previously used [32] in the studies of two-dimensional bosonic O(n) model. We are grateful to A. Tseytlin for drawing our attention to this fact.

where Δm is given by integral of $\Gamma(4\pi g i t)$ along the imaginary axis

$$\Delta m = -\frac{4g}{\pi}\,\mathrm{e}^{-\pi g}\int_0^{-i\infty}\frac{dt\ \mathrm{e}^{-4\pi g t - i\pi/4}}{t+\frac14}\big(\Gamma_+(4\pi g i t) + i\Gamma_-(4\pi g i t)\big) + \text{c.c.} \tag{A.2}$$

According to (2.16) the functions $\Gamma_\pm(4\pi g i t)$ have the following form

$$\Gamma_+(4\pi g i t) + i\Gamma_-(4\pi g i t) = V_0(4\pi g t) f_0(t) + V_1(4\pi g t) f_1(t)\,, \tag{A.3}$$

where f_0 and f_1 are defined as coefficients in front of V_0 and V_1, respectively, in the right-hand side of (2.16). They are given by a linear combination of the ratio of Euler gamma-functions

$$\begin{aligned} f_0(t) &= -\frac{\Gamma(\frac34)\Gamma(1-t)}{\Gamma(\frac34 - t)} + \frac{1}{4\pi g}\left[\left(\frac{3\ln 2}{4} + \frac{1}{8t}\right)\frac{\Gamma(\frac34)\Gamma(1-t)}{\Gamma(\frac34 - t)} - \frac{\Gamma(\frac14)\Gamma(1+t)}{8t\,\Gamma(\frac14 + t)}\right] + O(g^{-2})\,, \\ f_1(t) &= \frac{1}{4\pi g}\left[\frac{\Gamma(\frac14)\Gamma(1+t)}{4t\,\Gamma(\frac14 + t)} - \frac{\Gamma(\frac34)\Gamma(1-t)}{4t\,\Gamma(\frac34 - t)}\right] + O(g^{-2})\,. \end{aligned} \tag{A.4}$$

Notice that $f_1(t)$ is suppressed by factor $1/(4\pi g)$ compared to $f_0(t)$.

To work out the large g expansion of the integral (A.2) it is convenient to use the Mellin-Barnes representation for the functions $V_0(4\pi g t)$ and $V_1(4\pi g t)$. Using the integral representation (2.13) we find

$$V_0(4\pi g t)\,\mathrm{e}^{-4\pi g t} = \frac{\sqrt{2}}{\pi}\int_{-\delta-i\infty}^{-\delta+i\infty}\frac{dj}{2\pi i}\Gamma(-j)(4\pi g t)^j\int_{-1}^{1} du\,(1-u)^{j-1/4}(1+u)^{j+1/4} \tag{A.5}$$

and similar representation also exists for $V_1(4\pi g t)\,\mathrm{e}^{-4\pi g t}$. Their substitution into (A.3) and (A.2) yields after $u-$integration

$$\begin{aligned} \Delta m = -\frac{4\sqrt{2}g}{\pi^2}\,\mathrm{e}^{-\pi g}\,\Gamma(\tfrac14)\int_{-\delta-i\infty}^{-\delta+i\infty}\frac{dj}{2\pi i}\frac{\Gamma(-j)\Gamma(j+\frac34)}{\Gamma(j+2)}(2\pi g)^j\,\mathrm{e}^{-i\pi/4} \\ \times\int_0^{-i\infty}\frac{dt\,(4t)^j}{t+\frac14}\left[\frac12 f_0(t) + f_1(t)(j+1)\right] + \text{c.c.} \end{aligned} \tag{A.6}$$

To find the asymptotic expansion at large g we deform the integration contour in the complex j plane to the left and pick up the contribution of poles at negative j. These poles come from $\Gamma(j+\frac34)$ and from $t-$integral. Let us start with the latter ones.

By definition, $f_0(t)$ and $f_1(t)$ are real meromorphic functions of t with simple poles located at $t = \pm1, \pm2, \ldots$. Then, integration over small t produces simple poles located at negative integer j. However, due to the presence of $1/\Gamma(j+2)$, all these poles except $j=-1$ produce vanishing contribution to the $j-$integral. Calculating the residue at $j=-1$ we find

$$\Delta m = -\frac{4\sqrt{2}g}{\pi^2}\,\mathrm{e}^{-\pi g}\,\Gamma(\tfrac14)\Gamma(-\tfrac14)(2\pi g)^{-1}\,\mathrm{e}^{-i\pi/4}\,\frac12 f_0(0) + \text{c.c.} + \ldots = -\frac{8\sqrt{2}}{\pi^2}\,\mathrm{e}^{-\pi g} + \ldots\,, \tag{A.7}$$

where ellipses denote the contribution of poles produced by $\Gamma(j+\frac34)$ in (A.6). We notice that (A.7) cancels against similar term in the right-hand side of (A.1) and, therefore, m is determined by the contribution of poles at $j = -\frac34, -\frac74, \ldots$. Going through calculation of residues we obtain

$$m = \frac{4ig}{\pi^2}\Gamma(\tfrac34)(2\pi g)^{-3/4}\,\mathrm{e}^{-\pi g}\int_{-\delta-i\infty}^{-\delta+i\infty}\frac{dt\,(-t)^{-3/4}}{t+\frac14}\left[f_0(t) + \frac12 f_1(t) - (4\pi g t)^{-1}\frac{3}{32}f_0(t) + O(g^{-2})\right], \tag{A.8}$$

where the integration contour goes to the left from the branch cut that starts at $t = 0$. Calculating this integral and taking into account analytical properties of the functions (A.4) we find

$$m = -\frac{4g}{\pi}(2\pi g)^{-3/4}\,\mathrm{e}^{-\pi g}\,\Gamma(\tfrac{3}{4})2^{5/2}\left[f_0(-\tfrac{1}{4}) + \frac{1}{2}f_1(-\tfrac{1}{4}) + \frac{3}{32\pi g}f_0(-\tfrac{1}{4}) + O(g^{-2})\right], \tag{A.9}$$

leading to

$$m = (2\pi g)^{1/4}\,\mathrm{e}^{-\pi g}\,\frac{2^{1/2}}{\Gamma(\frac{5}{4})}\left[1 - \frac{6\ln 2 - 3}{32\pi g} + O(g^{-2})\right], \tag{A.10}$$

in an agreement with (2.17).

The strong coupling expansion of the mass scale (A.10) can be systematically improved by taking into account subleading $1/g$ corrections to the functions $f_0(t)$ and $f_1(t)$ in (A.4). Assuming that higher order corrections do not modify analytical properties of these functions and taking into account a contribution to (A.6) from an infinite sequence of poles at $j = -3/4 - n$ (with n nonnegative integer) we find after some algebra

$$m = -\frac{4g}{\pi}\,\mathrm{e}^{-\pi g}\,2^{5/2}\left[f_0(-\tfrac{1}{4})U_0^-(\pi g) + f_1(-\tfrac{1}{4})U_1^-(\pi g)\right]. \tag{A.11}$$

Here the notation was introduced for the functions

$$\begin{aligned} U_0^-(y) &= \int_0^\infty dt\ \mathrm{e}^{-2yt}\,t^{-1/4}(1+t)^{1/4} = (2y)^{-3/4}\Gamma(\tfrac{3}{4})\left[1 + \frac{3}{32y} + \dots\right], \\ U_1^-(y) &= \frac{1}{2}\int_0^\infty dt\ \mathrm{e}^{-2yt}\,t^{-1/4}(1+t)^{-3/4} = (2y)^{-3/4}\tfrac{1}{2}\Gamma(\tfrac{3}{4})\left[1 - \frac{9}{32y} + \dots\right], \end{aligned} \tag{A.12}$$

which can be expressed in terms of Whittaker functions of second kind

$$\begin{aligned} U_0^-(y) &= \tfrac{1}{2}\Gamma(\tfrac{3}{4})y^{-1}\,\mathrm{e}^{y}\,W_{1/4,1/2}(2y), \\ U_1^-(y) &= \tfrac{1}{2}\Gamma(\tfrac{3}{4})(2y)^{-1/2}\,\mathrm{e}^{y}\,W_{-1/4,0}(2y). \end{aligned} \tag{A.13}$$

Appendix B: Perturbative calculation of the free energy density

In this appendix we calculate the constant ($\mathcal{O}(\lambda^0)$) and quadratic ($\mathcal{O}(\lambda^2)$) terms in the perturbative expansion of the free energy (3.9).

The constant term sums up the quadratic fluctuations of fields $y(x)$ and $\boldsymbol{z}$ and it is given by a logarithm of the ratio of determinants of the kinetic operators:

$$\mathcal{F}^{(0)}(h) = \frac{n-2}{2V}\left[\mathrm{Tr}\log(-\partial^2 + M^2) - \mathrm{Tr}\log(-\partial^2 + \omega^2)\right], \tag{B.1}$$

with $M^2 = h^2 + \omega^2$ and V being the volume factor. Fortunately computing the difference also provides a regularization. By differentiating and integrating with respect to M and ω we can cast the result into the form

$$\mathcal{F}^{(0)}(h) = \frac{n-2}{2}\lim_{\omega^2\to 0}\int_{\omega^2}^{M^2} dm'^2 \int \frac{d^D p}{(2\pi)^D}\frac{1}{p^2 + m'^2}. \tag{B.2}$$

The momentum integration can be performed in the dimensional regularization with $D = 2 - \epsilon$ as

$$I(m) = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + m^2} = m^{D-2} \frac{\Gamma(1 - \frac{D}{2})}{(4\pi)^{\frac{D}{2}}} = \frac{m^{-\epsilon}}{4\pi} \left[\frac{2}{\epsilon} + \gamma + O(\epsilon) \right] , \tag{B.3}$$

with $\gamma = \Gamma'(1) + \ln(4\pi)$. As we will see in a moment, the same integral appears in higher order calculations. Using (B.3), we find for the constant term (B.2) (up to corrections vanishing as $\epsilon \to 0$)

$$\mathcal{F}^{(0)}(h) = \frac{n-2}{4\pi} h^{2-\epsilon} \left\{ \frac{1}{\epsilon} + \frac{\gamma}{2} + \frac{1}{2} \right\} . \tag{B.4}$$

The $\mathcal{O}(\lambda^2)$ term in the expansion (3.9) describes the two-loop correction to the free energy. It receives contribution from the last two terms inside the square brackets in (3.8)

$$\mathcal{F}^{(1)}(h) = \frac{1}{V} \int d^D x \, \langle \mathcal{L}_2(x) \rangle_0 - \frac{1}{2V} \int d^D x \int d^D x' \, \langle \mathcal{L}_1(x) \mathcal{L}_1(x') \rangle_0 \, , \tag{B.5}$$

with $\mathcal{L}_1(x)$ and $\mathcal{L}_2(x)$ defined in (3.6). Here the subscript '(0)' indicates that the expectation values are evaluated with the measure $\int \mathcal{D}y \mathcal{D}\boldsymbol{z} \exp\left(-\int d^D x \, \mathcal{L}_0(x)\right)$ (see Eqs. (3.8) and (3.6)).

The expression for $V^{-1} \int d^D x \, \langle \mathcal{L}_2(x) \rangle_0 = \langle \mathcal{L}_2 \rangle_0$ can be obtained in terms of the two VEV's

$$\langle y^2 \rangle_0 = I(\omega) \, , \qquad \langle \boldsymbol{z}^2 \rangle_0 = (n-2) I(M) \, . \tag{B.6}$$

Using translational invariance $\langle \partial^2 (y^4) \rangle_0 = 0$ and the equation of motion $\partial^2 y = \omega^2 y$, we get $\langle (y \partial_\mu y)^2 \rangle_0 = -\frac{\omega^2}{3} \langle y^4 \rangle_0 = -\omega^2 \langle y^2 \rangle_0^2$. Making use of similar relations for $\boldsymbol{z}-$field together with the factorization property $\langle y \partial_\mu y \boldsymbol{z} \partial_\mu \boldsymbol{z} \rangle_0 = \langle y \partial_\mu y \rangle_0 \langle \boldsymbol{z} \partial_\mu \boldsymbol{z} \rangle_0 = 0$, the contribution of $\mathcal{L}_2$ to (B.5) can be written in the following way:

$$\begin{aligned} \langle \mathcal{L}_2 \rangle_0 \;\; = \;\; & -\frac{\omega^2}{2} I^2(\omega) - \frac{(n-2) M^2}{2} I^2(M) \\ & + \frac{\omega^2}{8} \left[3 I^2(\omega) + 2(n-2) I(\omega) I(M) + 2(n-2) I^2(M) + (n-2)^2 I^2(M) \right] . \end{aligned} \tag{B.7}$$

We remove the infrared cut-off by taking the $\omega \to 0$ limit and obtain the relevant contribution from $\mathcal{L}_2$ as

$$\frac{1}{V} \int d^D x \, \langle \mathcal{L}_2(x) \rangle_0 = \langle \mathcal{L}_2 \rangle_0 = -\frac{n-2}{2} h^2 I^2(h) \, . \tag{B.8}$$

The contribution to the free energy (3.8) quadratic in $\mathcal{L}_1(x)$ has the form

$$\frac{1}{2V} \int d^D x \int d^D x' \langle \mathcal{L}_1(x) \mathcal{L}_1(x') \rangle_0 = \frac{1}{2} \int d^D x \, \langle \mathcal{L}_1(x) \mathcal{L}_1(0) \rangle_0 \, . \tag{B.9}$$

By writing

$$\mathcal{L}_1 = -ih (y^2 + \boldsymbol{z}^2) \partial_0 y = -ih \left(\tfrac{1}{3} \partial_0 y^3 + \boldsymbol{z}^2 \partial_0 y \right) , \tag{B.10}$$

we can see that the first term, being a total derivative, can be dropped. The remaining term gives rise to the diagrams shown on Fig. 2. The contribution of the first diagram on Fig. 2 involves the factor $\sim \int d^D x \, \langle \partial_0 y(x) \partial_0 y(0) \rangle_0$ with the integrand being a total derivative again.

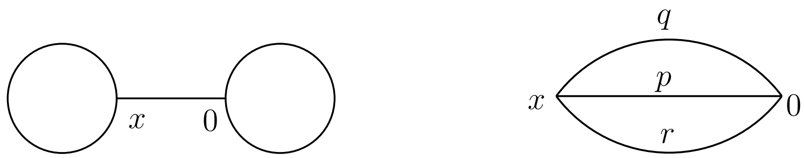

Figure 2: Two-loop diagrams contributing to (B.9). Solid lines denote two-dimensional scalar propagators.

Thus the right-hand side of (B.9) only receives contribution from the second diagram on Fig 2. Taking into account all possible contractions one finds

$$\frac{1}{2}\int d^Dx\langle\mathcal{L}_1(x)\mathcal{L}_1(0)\rangle_0 = -h^2(n-2)\int d^Dx\int\frac{d^Dq}{(2\pi)^D}\frac{\mathrm{e}^{iqx}}{q^2+M^2}\int\frac{d^Dp}{(2\pi)^D}\frac{p_0^2\,\mathrm{e}^{ipx}}{p^2+\omega^2}\int\frac{d^Dr}{(2\pi)^D}\frac{\mathrm{e}^{irx}}{r^2+M^2}\,, \tag{B.11}$$

where the factor $(n-2)$ counts the number of $\boldsymbol{z}-$fields circulating inside the loop. Doing the $p-$integration we observe that despite the fact that the integrand is not Lorentz covariant, the integral in the right-hand side of (B.11) could only depend on M^2 and ω^2 and, therefore, it should be Lorentz invariant. This allows us to simplify the $p-$integral as:

$$\int\frac{d^Dp}{(2\pi)^D}\frac{p_0^2\,\mathrm{e}^{ipx}}{p^2+\omega^2}\quad\Longrightarrow\quad\frac{1}{D}\int\frac{d^Dp}{(2\pi)^D}\frac{p^2\,\mathrm{e}^{ipx}}{p^2+\omega^2}=\frac{1}{D}\delta^{(D)}(x)+O(\omega\ln\omega)\,. \tag{B.12}$$

As a consequence, the $x-$integral in (B.11) becomes trivial and the remaining momentum integration gives $I^2(M)$. Then, taking the $\omega\to 0$ limit, we find from (B.11)

$$\frac{1}{2}\int d^Dx\,\langle\mathcal{L}_1(x)\mathcal{L}_1(0)\rangle_0 = -\frac{h^2}{D}(n-2)I^2(h)\,. \tag{B.13}$$

Substituting the relations (B.8) and (B.13) into (B.5) we derive the sought second order correction to the free energy density

$$\mathcal{F}^{(1)}(h) = h^2(n-2)I^2(h)\left[\frac{1}{D}-\frac{1}{2}\right] = \frac{n-2}{16\pi^2}h^{2-2\epsilon}\left\{\frac{1}{\epsilon}+\gamma+\frac{1}{2}\right\}\,. \tag{B.14}$$

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